International Winter School on Hydrology - 2019 Edition

Doctoral Winter School DATA RICH HYDROLOGY

Modelling **scaling** properties of precipitation fields

Part I

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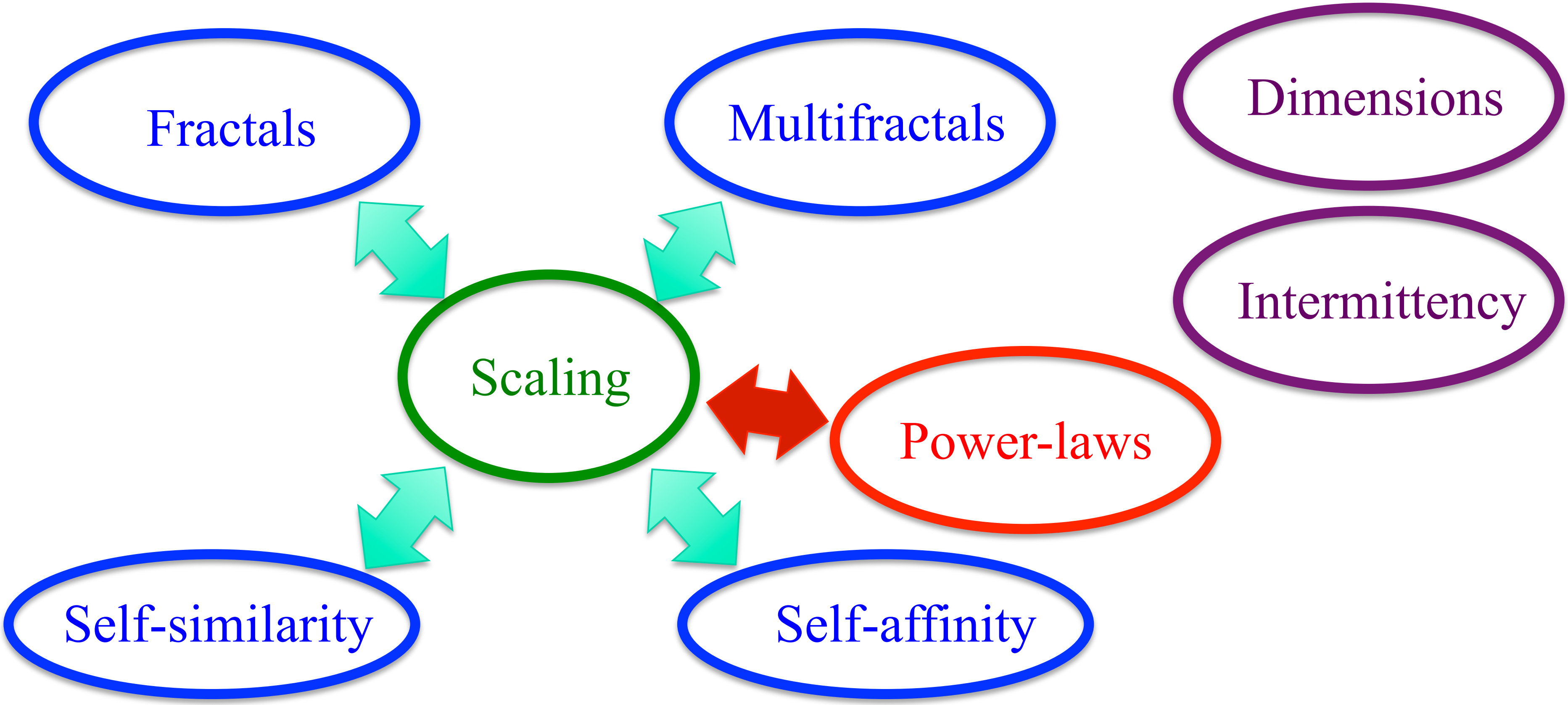
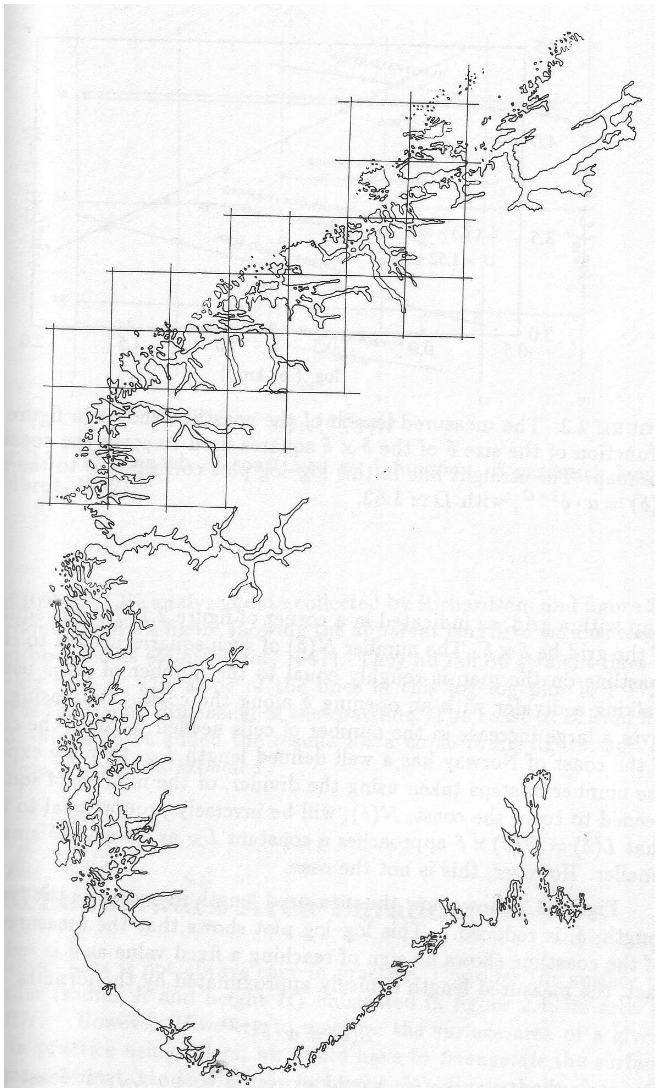
Some words and concepts often associated to scaling ….



RAW DEFINITION:

"Scaling" and more in general "scale invariance" refer to some kind of property which is observed/invariant across a range of scales ….
…. after proper transformation (e.g. self-similar or self-affine)

Let us start from fractals ….what is it?



Looking at a coastal line, we can observe some patterns which are repeated in different places!

If we make zoom in and out, we can observe similar patterns at smaller and larger scales!

… similarity ?? … scale-invariance ??

"Scaling" is often related to "fractals", that become very popular after Mandelbrot famous paper "*How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension*", Science, **1967**

What it happens if we measure the length of the coast line using smaller and smaller **rules**?

The smaller the ruler, the longer the coast … by zooming in we see more details!

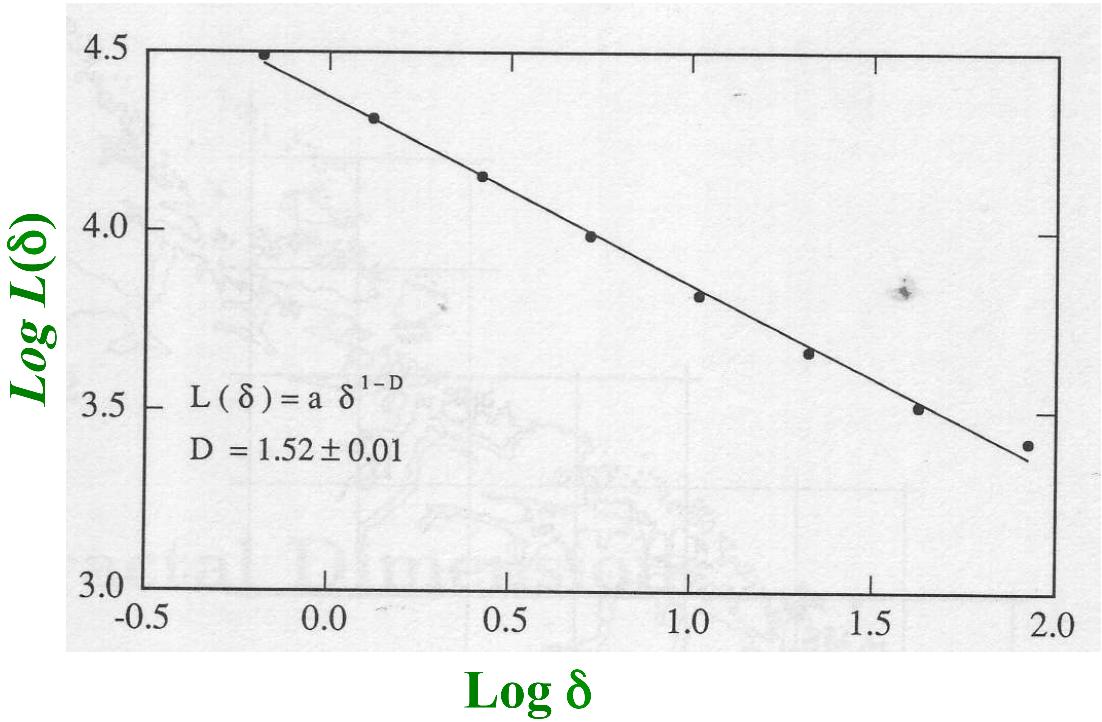
How long is a coastal line?

We are used to estimate the length of lines by taking a "ruler" of size δ and counting the number $N(\delta)$ of steps needed to move from one end of the line to the other:

- δ = size of the ruler
- $N(\delta)$ = number of steps to overlap the whole coastal line
- $L(\delta) = N(\delta)\,\delta$ = estimated length of the coastal line

Take a smaller and smaller ruler **… the smaller the ruler, the longer the coast!**

… by zooming, more details appear! … put the result in a log-log plot:



*Points are very close to a straight line with **negative slope**, i.e. our measure L(δ) follows a **power law**:*

$$L(\delta) = a\delta^{-b} \approx \delta^{-b}$$

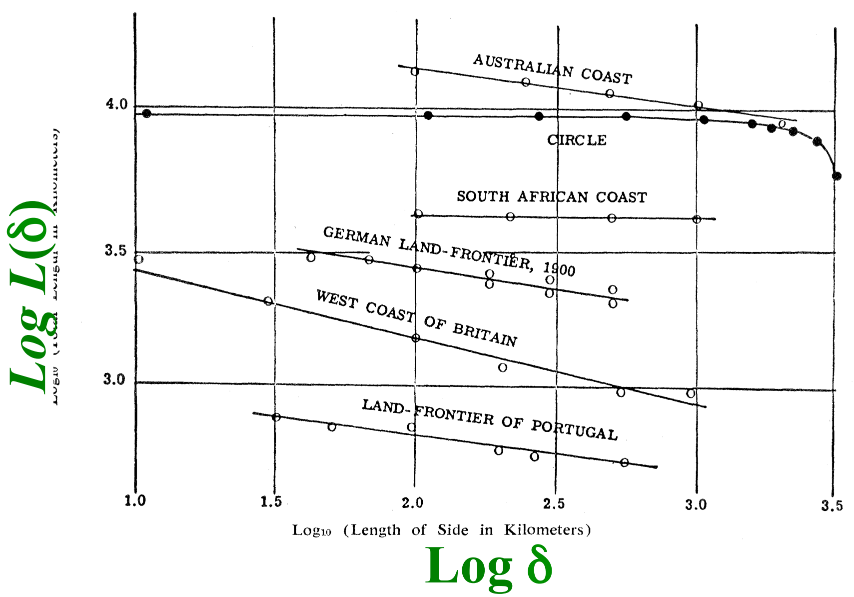
$$\log L(\delta) = \log a - b\log\delta$$

<u>There is a range of scaling!!</u>

How long is a coastal line?

Why our measure of the coast length increases when using smaller rulers?

For a straight line we expect $N(\delta) = L_T / \delta$ …. thus: $L(\delta) = N(\delta)\,\delta = L_T / \delta \times \delta = L_T$
Similarly, if our line has some roughness, but zooming in enough no new roughness appears, then we should expect the same result: i.e. for δ smaller than a certain threshold δ_0, if we take half of the ruler, the number of steps will double, … etc., so our estimation of the coastal length will be constant for any ruler size size $\delta<\delta_0$



Mandelbrot "*How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension*", Science, 1967

Note the result for the circle!

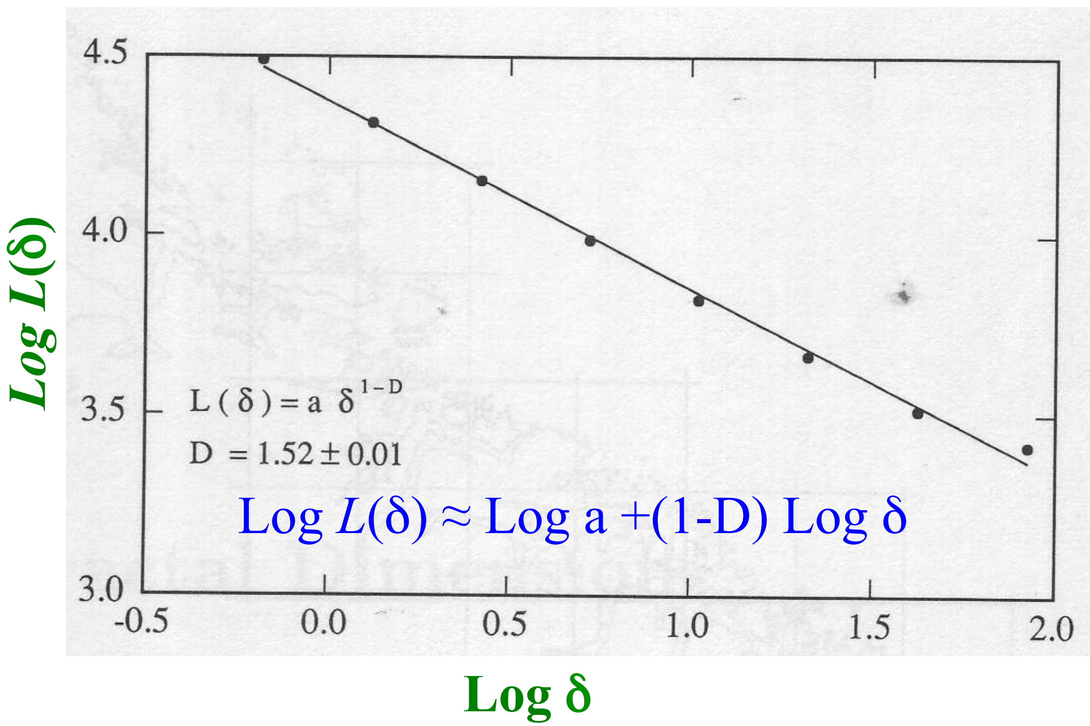
The circle is a regular line (with **topological dimension** $D_T = 1$)

Coastal lines are fractals, thus their **fractal dimension** is **larger** than the **topological dimension** of a regular line!

How long is a coast?

Why our measure of the coast length increases when using smaller rulers?

As noticed, for regular lines we expect $N(\delta) = L_T / \delta$ … so for half δ, $N(\delta)$ is double!
While for a fractal line, it happens that for **half** δ, $N(\delta)$ is **more than double**!
… thus for a fractal line $N(\delta) \approx 1 / \delta^D$
where D is a **fractal dimension**, larger than the **topological dimension D_T=1**!



A simple & practical approach:

The **fractal dimension** can be estimated in a log-log chart by the slope of the line in the range of **scale invariance**:

$$L(\delta) \approx N(\delta)\,\delta \approx a\,\delta^{-D}\,\delta = a\,\delta^{1-D}$$

….*or* … $N(\delta) \approx a\,\delta^{-D}$

$$\text{Log } N(\delta) \approx \text{Log } a - D \text{ Log } \delta$$

…in our case: D = 1.5

Restore previous question….what is a fractal?

Mandelbrot tried to give some formal definitions of **fractals**:

"*A fractal is by definition a set for which the* ***Hausdorff-Besicovitch dimension*** *strictly exceeds the topological dimension*" (Mandelbrot, 1982)

… some years later Mandelbrot decided to give a simpler definition…

"*A fractal is a shape made of parts similar to the whole in some way*" (Mandelbrot, 1986)

…There is not a common definition of fractal, but we can find different definitions of ***fractal dimension***, which represents the main property of a fractal set:

- Box-counting dimension *(most popular and easy to apply)*
- Hausdorff-Besicovitch dimension
- (self-)similarity dimension

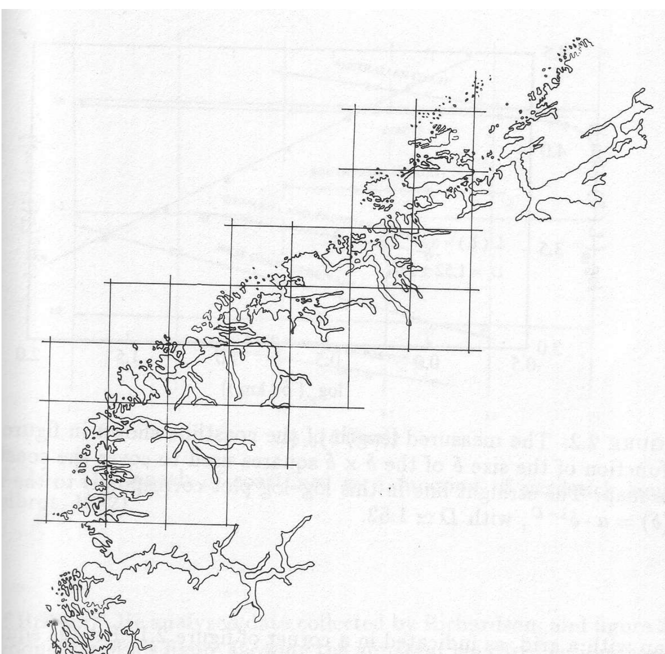
Box-counting dimension D_B

Concepts of *box-counting* were applied by Kolmogorov when studying turbulence in early 1930s and sometimes referred to as *capacity dimension*.

Box-counting dimension D_B of a fractal set $S \subset R^n$ where R^n is a n-dimensional embedding space:

$$D_B = \lim_{\delta \to 0} \frac{\log N(\delta)}{\log(1/\delta)}$$

where $N(\delta)$ is the minimum number of boxes/cubes/ipercubes (or circles/spheres/iperspheres) of maximum size δ which can entirely cover the fractal set S



The line is embedded in a plane (R^2):

- $n = 2$ (dimension of the embedding space)
- $D_T = 1$ (topological dimension)
- D_B = fractal dimension by box-counting <n

For practical applications we look for ranges of scale invariance in the log-log plane:

$$\text{Log } N(\delta) \approx - D_B \text{ Log } \delta$$

Hausdorff-Besicovitch dimension D_H

… I will give here a simplified definition based on the observation that the number of cubes of side δ needed to cover a fractal set scales as $N(\delta) \approx 1 / \delta^D$

The Hausdorff-Besicovitch dimension D_H is that value of d that makes the following limit <u>finite</u>:

$$\lim_{\delta \to 0} H_d(\delta) = \lim_{\delta \to 0} N(\delta)\delta^d = \begin{cases} 0 & d > D_H \\ \mathit{finite} & d = D_H \\ \infty & d < D_H \end{cases}$$

where $H_d(\delta)$ is the *Hausdorff measure* and d is like a testing exponent.

It is straightforward to show that the condition for the above limit to be finite is that $N(\delta)$ must be scaling as $N(\delta) \approx 1/\delta^{D_B}$, such that:

$$H_d(\delta) = N(\delta)\delta^d \approx \delta^{(d-D_B)}$$

…. more formal definitions: $H_d(\delta) = \min \sum_i \delta_i^d$ where $\delta_i < \delta$ thus $D_H \leq D_B$

Self-similarity dimension D_S

Self-similarity

Let us consider an **isotropic transformation** $S' = r(S)$ that maps points $x \in S$ into other points $x' = rx$, where $r < 1$ is a contracting factor and $x' \in S$

The set S is said to be **self-similar** with respect the transformation $r(S)$, if the original set S can be entirely covered without overlapping by $m(r)$ replies of S'

Self-similarity implies that there exists a range of scales (***range of self-similarity***) where the following relation is valid:

$$N(r\delta) = m(r)\, N(\delta)$$

- $N(r\delta)$ = number of boxes of side $r\delta$ needed to cover the entire original set S
- $N(\delta)$ = number of boxes of side δ needed to cover the entire original set S
- $m(r)$ = number of replies S' needed to reproduce the entire original set S

Thus, in the range of self-similarity we should observe the **power law** $N(\delta) \approx \delta^{-D_S}$

Indeed substituting above $N(r\delta) \approx r^{-D_S}\delta^{-D_S}$ we obtain $m(r) \approx r^{-D_S}$ and then …

..the **self-similarity dimension:** $D_S = \dfrac{\log m(r)}{\log(1/r)}$

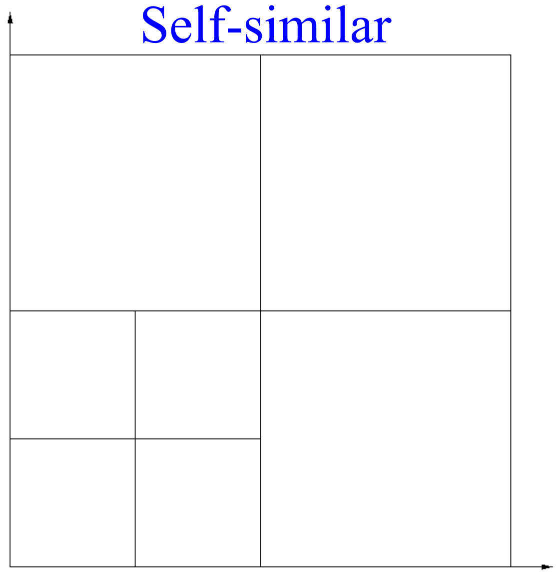
Self-affinity

Self-similar fractals are invariant under some **isotropic transformations** *S'* = *r*(*S*) where the contracting factor r < 1 is a **scalar**!

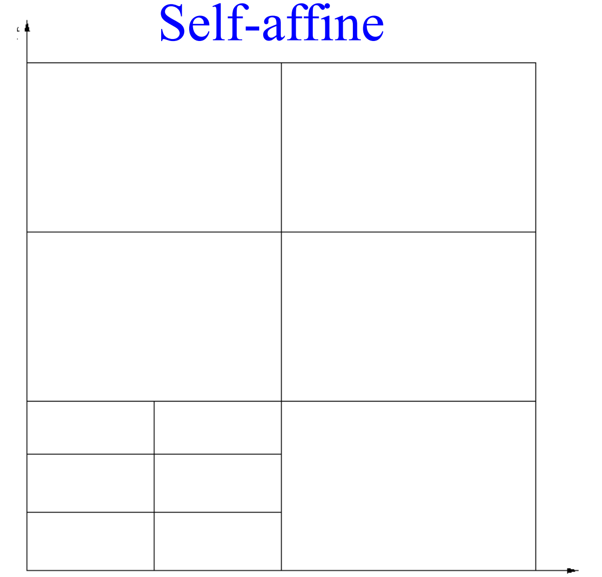
Self-affine fractals are invariant under **anisotropic transformations** *S'* = ***r***(*S*) that maps the original points $x \in S$ into other points $x' \in S$ with different contracting factor in each dimension of the embedding space and also rotation.

In general, a different number of (rotated) replies *S*' can be needed in each direction to reproduce the entire original set S (**r** is a tensor). A simple example:

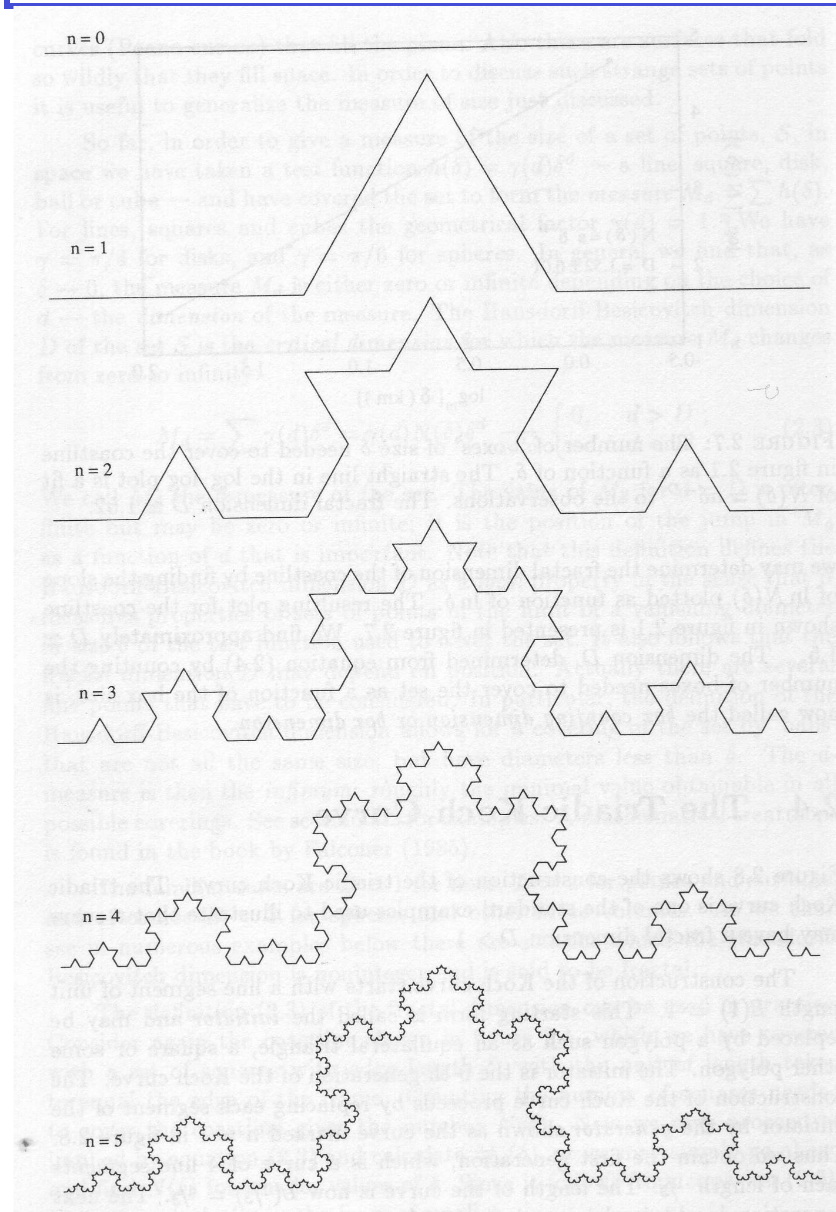
Self-similar



Self-affine



Example: the triadic Koch curve



$n = 0 \quad \delta = 1 \quad N(\delta) = 1$

$n = 1 \quad \delta = 1/3 \quad N(\delta) = 4$

$n = 2 \quad \delta = 1/3^2 \quad N(\delta) = 4^2$

$n = k \quad \delta = 1/3^k \quad N(\delta) = 4^k$

Box-counting dimension:

$\text{Log}\,\delta = -k\,\text{Log}\,3 \rightarrow k = \text{Log}(1/\delta) / \text{Log}\,3$

$N(\delta) = 4^{\text{Log}(1/\delta) / \text{Log}\,3} \rightarrow$

$\text{Log}\,N(\delta) = (\text{Log}\,4 / \text{Log}\,3)\,\text{Log}\,(1/\delta)$

$$D_B = \lim_{\delta \to 0} \frac{\log N(\delta)}{\log(1/\delta)} = \frac{\log 4}{\log 3} = 1.2628....$$

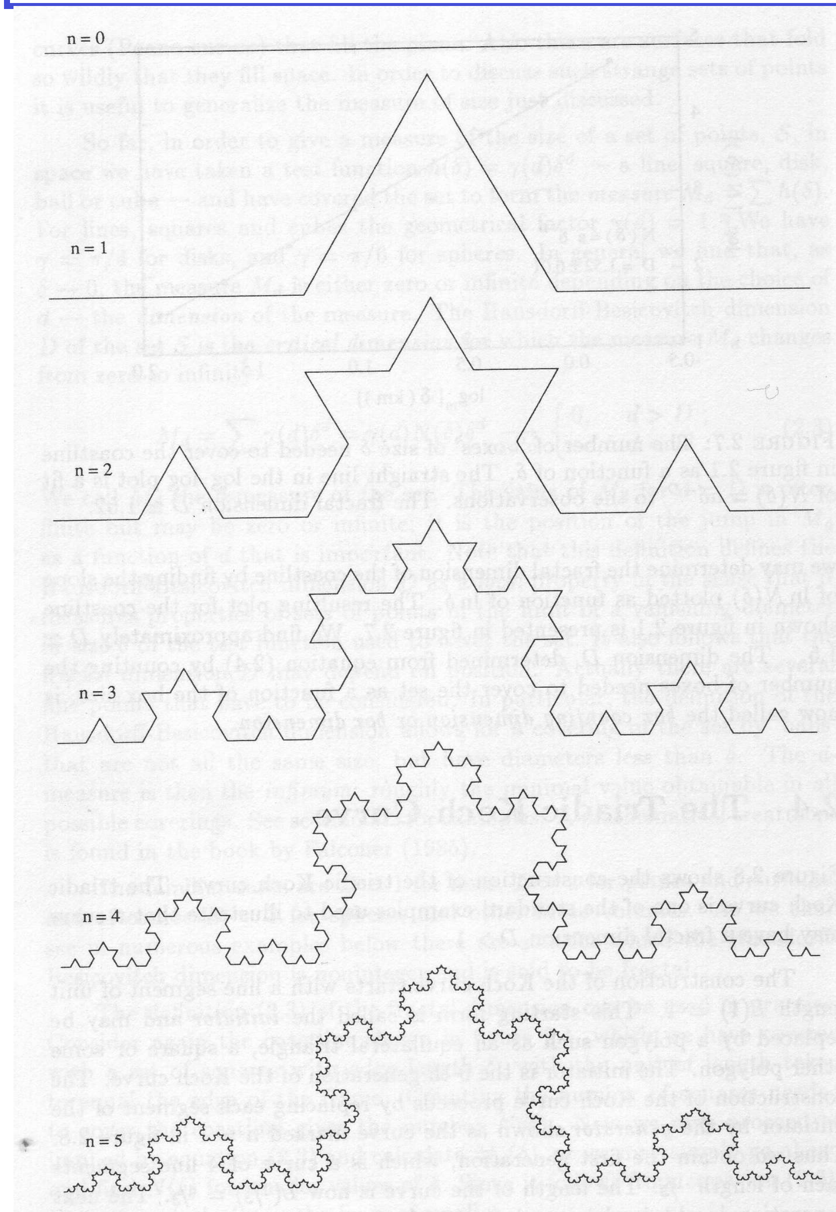
Self-similarity dimension:

$r = 1/3 \quad \rightarrow \text{Log}\,(1/r) = \text{Log}\,3$

$m(r) = 4 \quad \rightarrow \text{Log}\,m(r) = \text{Log}\,4$

$$D_S = \frac{\log m(r)}{\log(1/r)} = \frac{\log 4}{\log 3} = 1.2628....$$

Example: the triadic Koch curve (cont.)



Hausdorff-Besicovitch dimension:

$$\text{Log } N(\delta) = (\text{Log } 4 / \text{Log } 3) \text{ Log } (1/\delta)$$

$$N(\delta) \approx \delta^{-(\text{Log } 4 / \text{Log } 3)}$$

$$H_d(\delta) = N(\delta)\delta^d = \delta^{d-(\log 4/\log 3)}$$

$$\lim_{\delta \to 0} H_d(\delta) = \lim_{\delta \to 0} N(\delta)\delta^d = \begin{cases} 0 & d > D_H \\ \textit{finite} & d = D_H \\ \infty & d < D_H \end{cases}$$

$$D_H = \text{Log } 4 / \text{Log } 3 = 1.2628\ldots$$

From fractals to multifractals

Fractal geometry characterizes (by a topological point of view) sets $S \subset R^n$ through their *fractal dimension* and possible *self-similar* or *self-affine* properties.
When some kind of measure is distributed on this fractal set S, we need **multifractal theory** to proper describe such a system.

Fractals

Fractals describe complex geometries through different scales.

Multifractals

Miltifractals describe heavy tailed probability distributions of measures through different scales, which can be distributed over a fractal set (or not, it is not a mandatory condition).
A feature of multifractal measures is that they fluctuate from point to point and their intensity can change a different scales (**intermittency** ... which is not only a on/off process).

Support of multifractal measures

Let $S \subset R^n$ be a fractal set of fractal dimension D_S where a mass/variable/field $\phi(\mathbf{x})$ is unevenly or randomly distributed.
S is said the support of our measure (the condition to be a fractal set is not mandatory, but we keep it for generality).

Without loss of generality we assume:

- $\phi(\mathbf{x})$ is null in the complement in R^n of the set S
- the integral of $\phi(\mathbf{x})$ in R^n is unitary.

Singular exponents and multifractal spectrum

We can then introduce an integral measure of $\phi(\mathbf{x})$ in each n-dimensional volume $B_i(\delta)$ of size δ centred in the i-th position:

$$\mu_i(\delta) = \int_{B_i(\delta)} \phi(\mathbf{x}) d\mathbf{x} \tag{1}$$

where $\mu_i(\delta)$ is then referred to as **multifractal measure** if we can observe the following limit (where α are referred to as **singularity exponents**):

$$\lim_{\delta \to 0} \mu_i(\delta) \sim \delta^{\alpha} \tag{2}$$

A main feature of multifractals is that α fluctuates from point to point. We can thus introduce a probability distribution of the volumes where the limit (2) holds, or we can introduce the number $N_\alpha(\delta)$ of volumes $B_i(\delta)$ where the measure $\mu_i(\delta)$ follows the power law (2):

$$N_\alpha(\delta) \sim \delta^{-f(\alpha)} \tag{3}$$

where $f(\alpha)$ is the **multifractal spectrum**.

Partition functions

Using (2) and (3) we can evaluate the **partition functions** $Z_q(\delta)$, i.e. the sum of the q-order moments of our measure $\mu_i(\delta)$ in (1):

$$Z_q(\delta) = \sum_i \mu_i(\delta)^q \sim \int \delta^{q\alpha} \delta^{-f(\alpha)} d\alpha \sim \delta^{\tau(q)} \tag{4}$$

where the exponent $\tau(q)$ can be derived by a *saddle point integration*. Indeed, for small δ, the main contribution in the above integral comes from those α values making small the exponent $q\alpha - f(\alpha)$:

$$\tau(q) = \min_{0<\alpha<\infty} [q\alpha - f(\alpha)] \tag{5}$$

... thus for any q we nullify the derivative with respect α and obtain:

$$q = \left. \frac{df(\alpha)}{d\alpha} \right|_{\alpha(q)} \tag{6}$$

Multifractal measures are characterized by a non linear behaviour of exponents $\tau(q)$ as a function of q.

Partition functions and multifractal spectra

For any q, from (6) we can derive a relation $\alpha = \alpha(q)$ that can be substituted in (5), and finally we obtain $\tau(q)$ as a function of the singularity exponent α and the multifractal spectrum $f(\alpha)$:

$$\tau(q) = q\alpha(q) - f[\alpha(q)] \tag{7}$$

.... and with some more mathematics we can obtain the singularity exponent α and the multifractal spectrum $f(\alpha)$ as a function of $\tau(q)$:

$$\alpha(q) = \frac{d\tau(q)}{dq} \tag{8}$$

$$f[\alpha(q)] = q\frac{d\tau(q)}{dq} - \tau(q) \tag{9}$$

In conclusion, the two representations $\tau(q)$ and $f(\alpha)$ are equivalent, and we can switch from one to the other!

A typical multifractal spectum

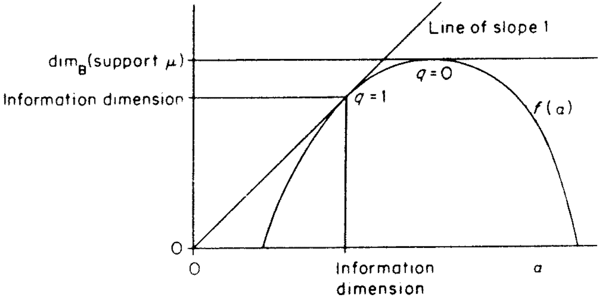


Figure 17.2 Features of the multifractal spectrum—the graph of $f(\alpha)$ against α

Support dimension: $q = 0$; $\tau(0) = -D_S$; $\left.\frac{df(\alpha)}{d\alpha}\right|_{\alpha(0)} = 0$ (i.e. f is max)

Information dimension (entropy): $q = 1$; $\tau(1) = 0$; $f[\alpha(1)] = \alpha(1)$

Structure functions

Structure functions are the expected values of the q-order moments of our integral measures (other uses mean intensity) of $\phi(\mathbf{x})$ at scale δ:

$$S_q(\delta) = < \left[\underbrace{\int_{B(\delta)} \phi(\mathbf{x}) d\mathbf{x}}_{\mu(\delta)} \right]^q > = \frac{1}{N(\delta)} \sum_i \mu_i(\delta)^q \qquad (10)$$

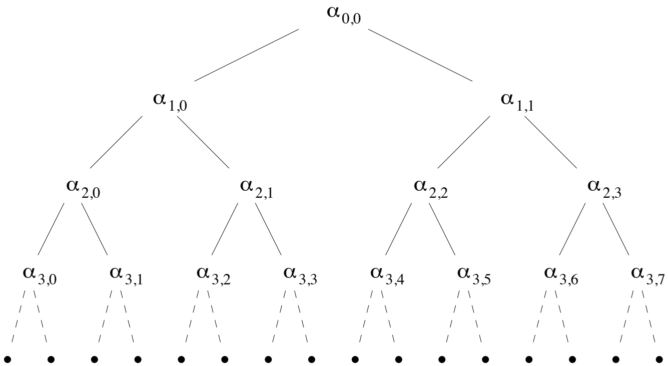
where $< \cdot >$ is an average operator on all $N(\delta)$ non-overlapping n-dimensional volumes $B(\delta)$, which are needed to completely cover the sub-space in R^n embedding our mass/variable/field $\phi(\mathbf{x})$: thus $N(\delta) \sim \delta^{-n}$.
It is easy to derive:

$$S_q(\delta) \sim \delta^n \delta^{\tau(q)} = \delta^{\zeta(q)} \qquad (11)$$

$$\zeta(q) = n + \tau(q) \qquad (12)$$

... as well $\zeta(0) = n - D_S$ and $\zeta(1) = n$

A discrete random cascade in R with branching number 2



Cascade starts from an initial value α_0.
At each bifurcation, two son tiles with values $\alpha_{j,k}$ are generated by multiplying the father value by a i.i.d. random variable η (**generator**).
The first index j is the fragmentation level, while $k = 0, ..., 2^j - 1$ is the position.
Note that other integer branching numbers (3,4,5 ...) can be used.

Random cascades in R

Let us assume that our signal $\phi(\mathbf{x})$ is generated in the interval $x \in [0,1]$. At j-th fragmentation level, the signal is partitioned into $N(\delta) = 2^j$ intervals of side $\delta = 1/2^j$, thus $j = -\log_2 \delta$.
The value of each cascade tile $\alpha_{j,k} = \alpha_{j-1,k/2} * \eta$ is assumed to be the **integral measure** of our desired field $\phi(\mathbf{x})$ at scale δ:

$$\alpha_{j,k} = \int_{k\delta}^{(k+1)\delta} \phi(\mathbf{x}) d\mathbf{x}$$

Under these hypotheses, we want to determine if the partition functions scales with δ:

$$S_q(\delta) = < \left[\int_\delta \phi(\mathbf{x}) d\mathbf{x} \right]^q > \sim \delta^{\zeta(q)}$$

... and if the exponents $\zeta(q)$ are a nonlinear function of q.
In such a case the generated signal is multifractal.

Random cascades in R

Since the generator η is a i.i.d. random variable, at each j-th fragmentation level the expectation of the q-moment of any integral measure α is the same regardless the position $k = 0, ..., 2^j - 1$:

$$\overline{\alpha_{j,k}^q} = \overline{\alpha_j^q} = \alpha_0^q \overline{\eta^q}^j \tag{13}$$

indeed:

$$\overline{\alpha_j^q} = \alpha_0^q \int \eta_1^q \cdots \eta_j^q p(\eta_1, \cdots, \eta_j) d\eta_1 \cdots d\eta_j =$$
$$\alpha_0^q \int d\eta_1 \cdots \int d\eta_j \eta_1^q \cdots \eta_j^q p_1(\eta_1) \cdots p_j(\eta_j) = \alpha_0^q \left[\int \eta^q p(\eta) d\eta\right]^j = \alpha_0^q \overline{\eta^q}^j$$

We want the integral I of our signal to be 1:

$$I = \int_0^1 \phi(x) dx = \sum_{k=0}^{2^j-1} \alpha_{j,k} = 2^j \overline{\alpha_j} = 2^j \alpha_0 \overline{\eta}^j = \alpha_0 (2\overline{\eta})^j$$
$$\alpha_0 = (2\overline{\eta})^{-j}$$

Random cascades in R

$$S_q(\delta) = < \left[\int_\delta \phi(\mathbf{x}) d\mathbf{x}\right]^q > = \overline{\alpha_j^q} = \alpha_0^q \overline{\eta^q}^j = (2\overline{\eta})^{-jq} \overline{\eta^q}^j = \left[(2\overline{\eta})^q \overline{\eta^q}^{-1}\right]^{-j}$$

$$\log_2 S_q(\delta) = -j \ \log_2 \left[(2\overline{\eta})^q \overline{\eta^q}^{-1}\right] = \log_2 \delta \ \log_2 \left[(2\overline{\eta})^q \overline{\eta^q}^{-1}\right]$$

$$S_q(\delta) = \delta^{\log_2 \left[(2\overline{\eta})^q \overline{\eta^q}^{-1}\right]}$$

$$\zeta(q) = \log_2 \left[(2\overline{\eta})^q \overline{\eta^q}^{-1}\right] = q(1 + \log_2 \overline{\eta}) - \log_2 \overline{\eta^q}$$

Now we assume that our signal $\phi(\mathbf{x})$ is generated in $\mathbf{x} \in [0,1]^n$.
At j-th fragmentation level, the signal is partitioned into $N(\delta) = 2^{nj}$ n-dimensional volumes of side $\delta = 1/2^j$
Imposing that the integral of $\phi(\mathbf{x})$ in $\mathbf{x} \in [0,1]^n$ is 1 we obtain:

$$\alpha_0 = (2^n \overline{\eta})^{-j}$$

... we can show that multifractal exponents $\zeta(q)$ are expected to be:

$$\zeta(q) = q(n + \log_2 \overline{\eta}) - \log_2 \overline{\eta^q}$$

Important note

Although previous and following results are derived by structure functions defined through **integral measures** of $\phi(\mathbf{x})$ at different scales, the same scaling properties can be derived using **average measures** of $\phi(\mathbf{x})$ through scales. In the latter case previous structure functions must be divided by δ^n and we found a slight different expression for multifractal exponents, but it is easy to switch from one framework to the other.

Log-Poisson generator η

$$\eta = \beta^y$$

where β is a constant, while y is a i.i.d. random variable following a Poisson distribution with parameter c:

$$P(y = m) = \frac{c^m e^{-c}}{m!}$$

We can now derive any q-moment of the generator η:

$$\overline{\eta^q} = \overline{\beta^{qy}} = \sum_{m=0}^{\infty} \beta^{qm} \frac{c^m e^{-c}}{m!} = \exp\left[c\left(\beta^q - 1\right)\right]$$

$$\overline{\eta} = \exp\left[c\left(\beta - 1\right)\right]$$

... and finally a closed form for expected multifractal exponents:

$$\zeta(q) = qn + c\,\frac{q\left(\beta - 1\right) - \left(\beta^q - 1\right)}{\ln 2}$$

By tuning only 2 parameters (c, β) we can generate discrete random cascades that very closely reproduce observed multifractality.

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Modelling **scaling** properties of precipitation fields

Part III

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Università di Cagliari

Perugia (Italy), January 28 – February 1, 2019 – Villa Colombella

Introduction to space-time rainfall downscaling problems

MOTIVATION:

*There was a need to bridge the gap between the **large scales** resolved by NWP models and the **small scales** required by hydrological modelling.*

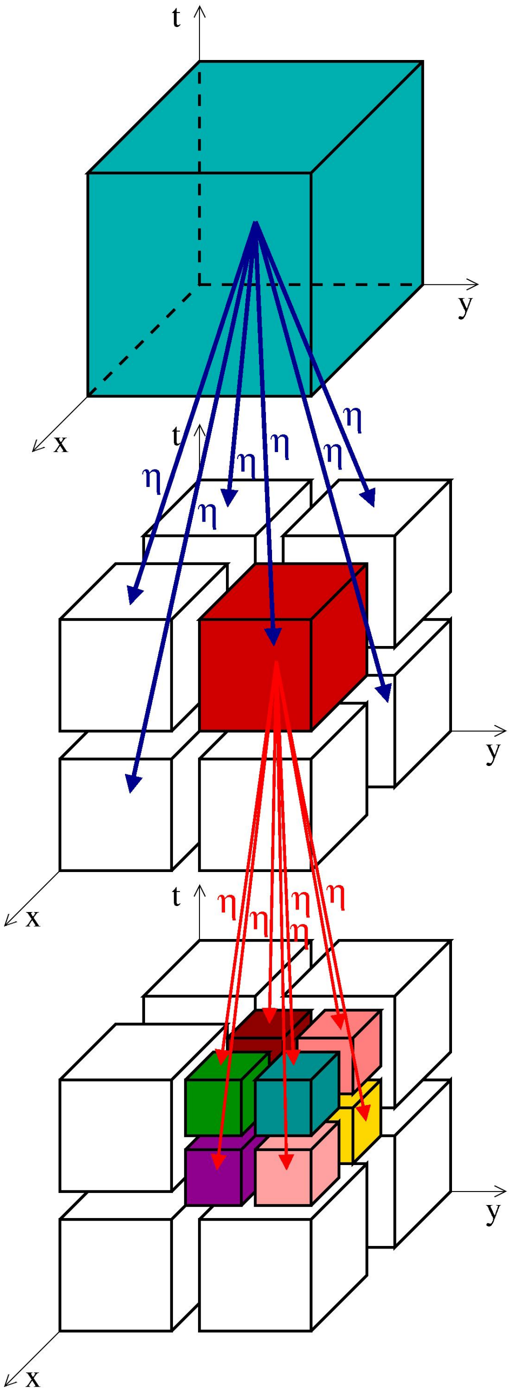
(coupling meteorological and hydrological models working on different space-time grid resolution)

EVIDENCE:

*Rainfall fields display **fluctuations in space and time** that increase as the scale of observation decreases.*

METODOLOGY

*Multifractal theory represents a solid base to characterize scale-invariance properties observed in rainfall fields as well as to develop downscaling models able to reproduce observed statistics (e.g. **multifractal cascades**).*



Two questions in space-time rainfall downscaling problems

Is there a relationship between space and time scales where we can observe the same statistical properties?

- **Self-similarity (i.e. scale isotropy)**

  or

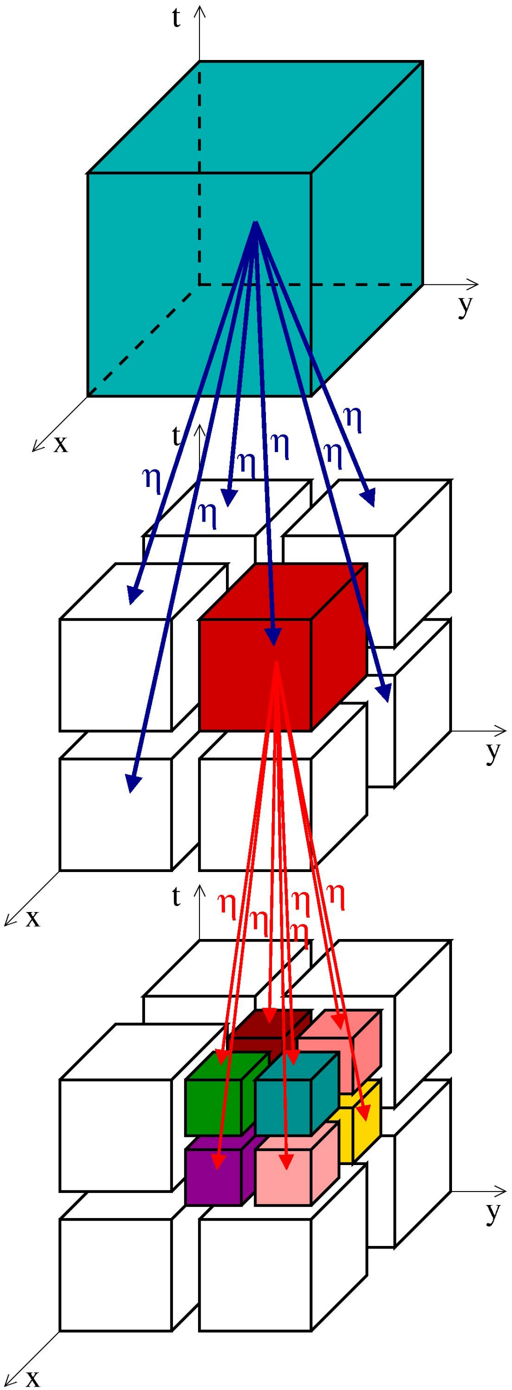
  Self-affinity (i.e. scale anisotropy)

Is the probability distribution of rain rate the same in each point/grid-cell (x,y)?

- **Space homogeneity**

  or

  heterogeneity (e.g., due to orography)



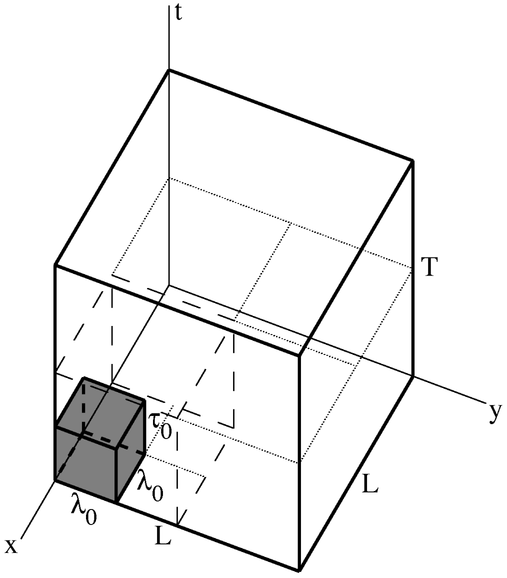
Multifractal analysis of space-time rainfall fields

HYPOTHESES:

- **space-time self-similarity** ($\tau = \lambda/U$)

  λ = space scale *linear relation*
  τ = time scale
  U = const. ratio of space and time scales

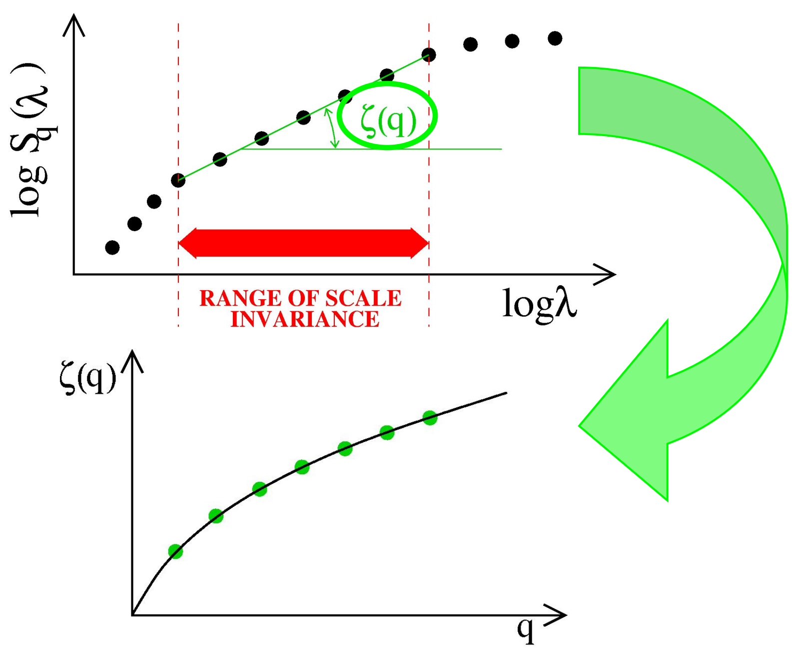
- **space homogeneity**



$$\mu_{i,j,k}(\lambda) = \int_{x_i}^{x_i+\lambda} dx \int_{y_j}^{y_j+\lambda} dy \int_{t_k}^{t_k+\lambda/U} dt \;\; i(x,y,t)$$

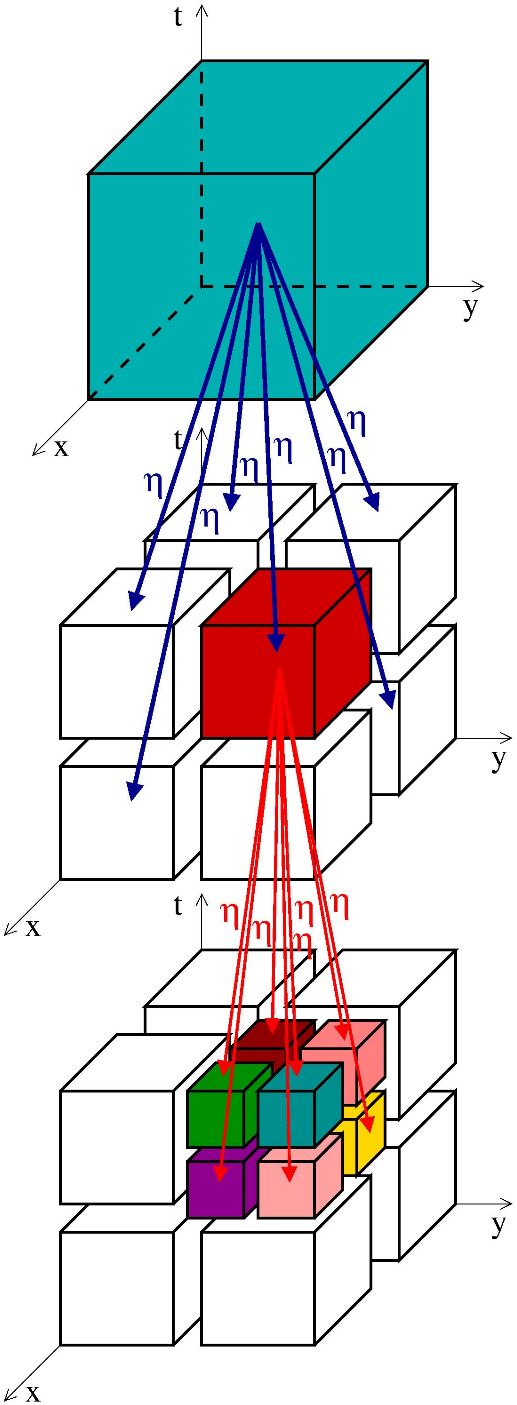
Partition function & scale invariance:

$$S_q(\lambda) = \left\langle \mu_{i,j,k}(\lambda)^q \right\rangle \approx \lambda^{\zeta(q)}$$



Multifractal measures:

$\zeta(q)$ non-linear function of q



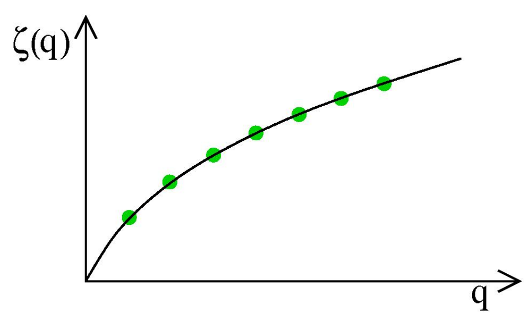
Rainfall downscaling with random cascades: the STRAIN model

R. Deidda, R. Benzi, F. Siccardi (1999), *Water Resources Research*, **35**
R. Deidda (2000), *Water Resources Research*, **36**

Log-Poisson generator: $\eta = \beta^{y}$
where y is a Poisson distributed i.i.d. random variable with parameter (average) c

The theoretical expectation for multifractal exponents is:

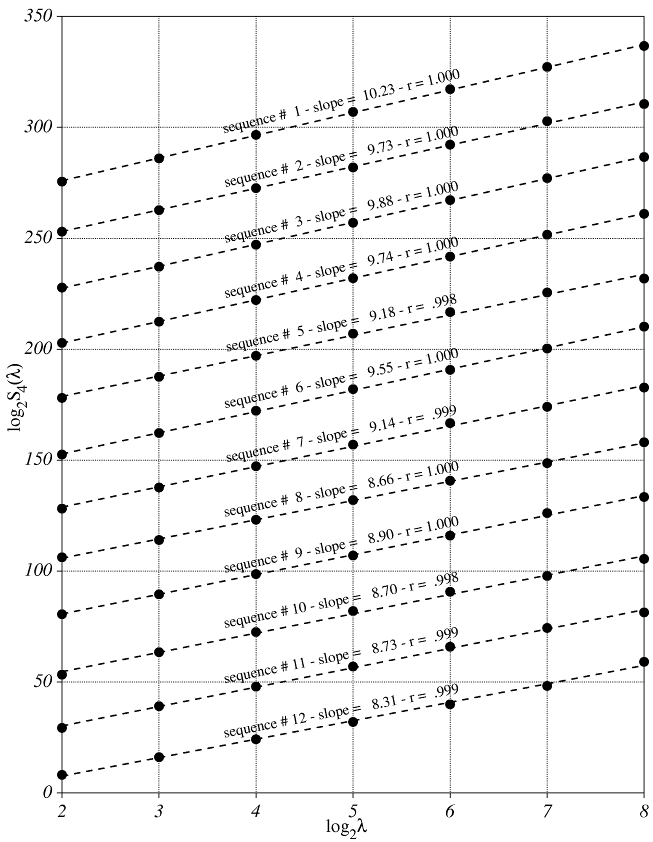
$$\zeta(q) = 3q + \frac{c}{\ln 2}\left[\left(1-\beta^{q}\right) - q\left(1-\beta\right)\right]$$

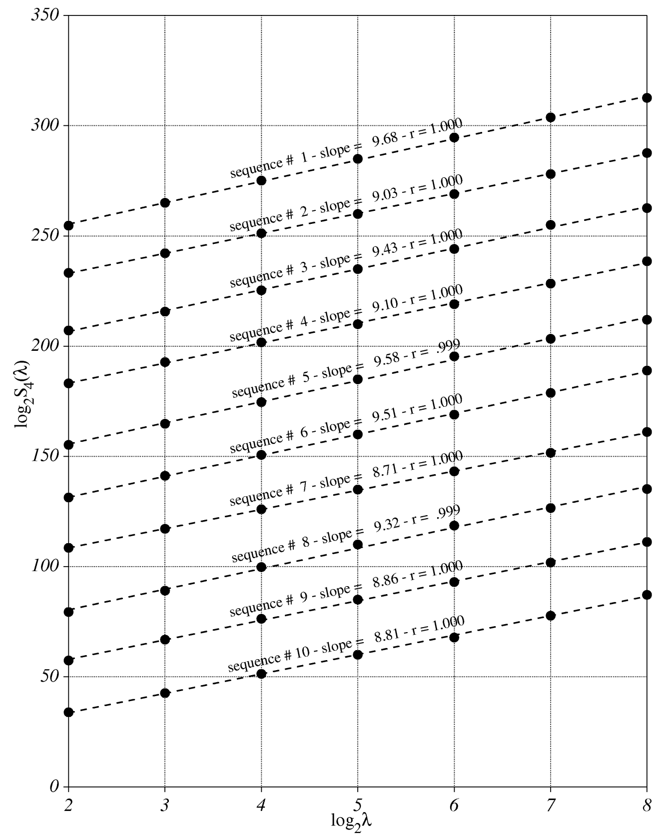


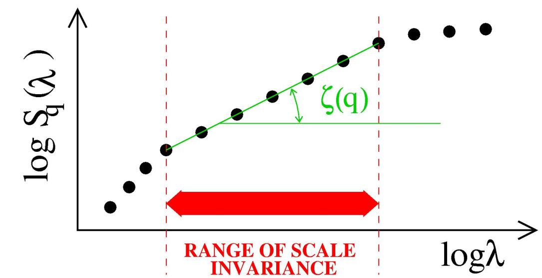
Best fit procedure to estimate c & β

GATE: partition functions

$$S_q(\lambda) = \left\langle \mu_{i,j,k}(\lambda)^q \right\rangle \approx \lambda^{\xi(q)}$$

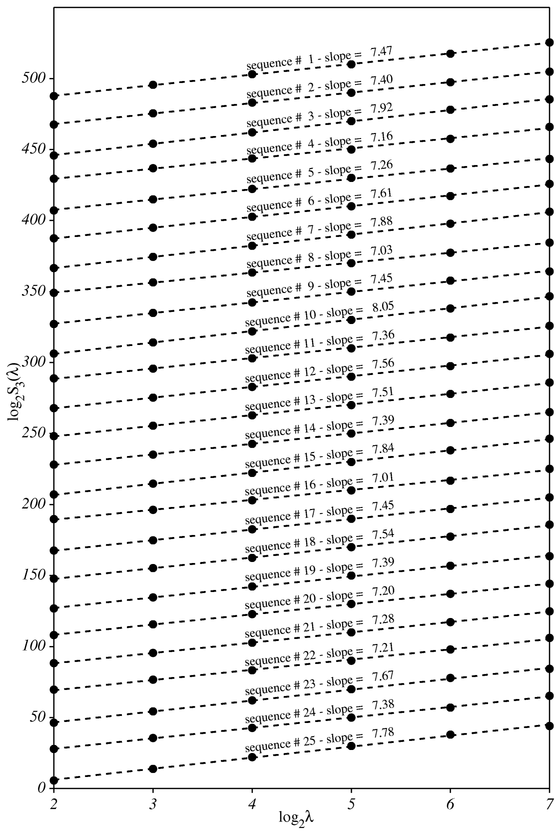


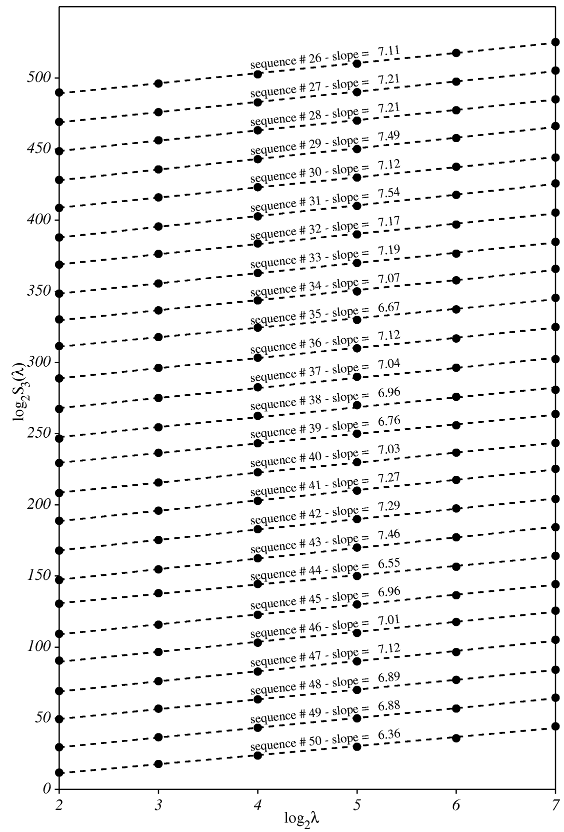


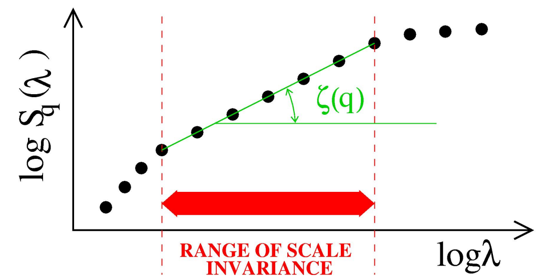


Log-log plot of fourth-order partition functions $S_4(\lambda)$ versus λ scales ranging from $\lambda_0 = 4$ km to $L = 256$ km (time scales range from 15 minutes to 16 hours).

TOGA: partition functions $S_q(\lambda) = \left\langle \mu_{i,j,k}(\lambda)^q \right\rangle \approx \lambda^{\xi(q)}$

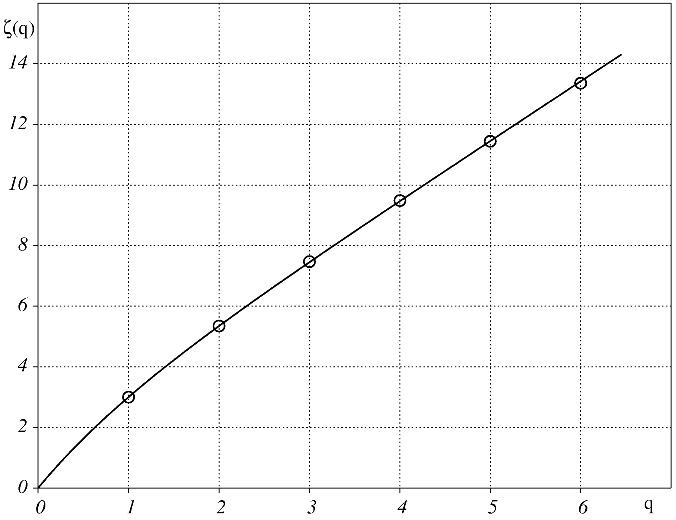


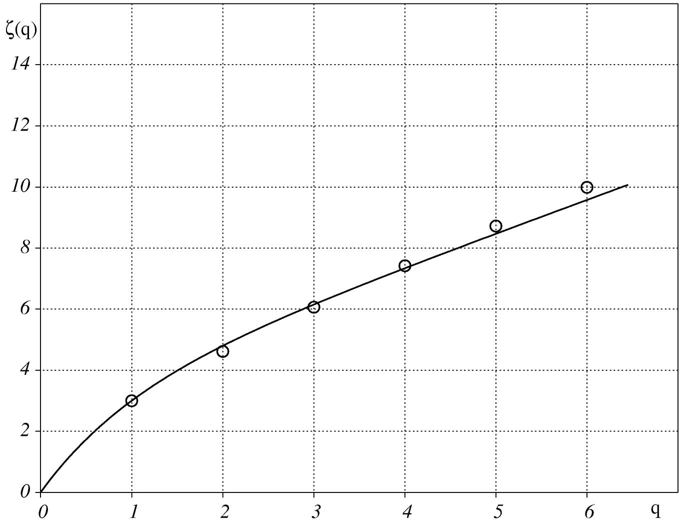


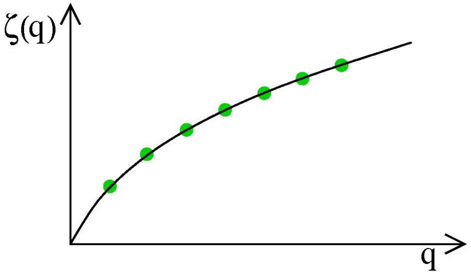


Log-log plot of third-order partition functions $S_3(\lambda)$ versus λ scales ranging from $\lambda_0 = 4$ km to L = 128 km (time scales range from 10 minutes to 5h:20’).

Estimates of multifractal exponents on two sequences (high and low rain rate)







Calibration of the STRAIN model (cascade generator $\eta = \beta^y$, where y is a Poisson distributed random variable with mean c).

The c and β model parameters can be estimated fitting sample to expected MF exponents ζ(q) on each sequence:

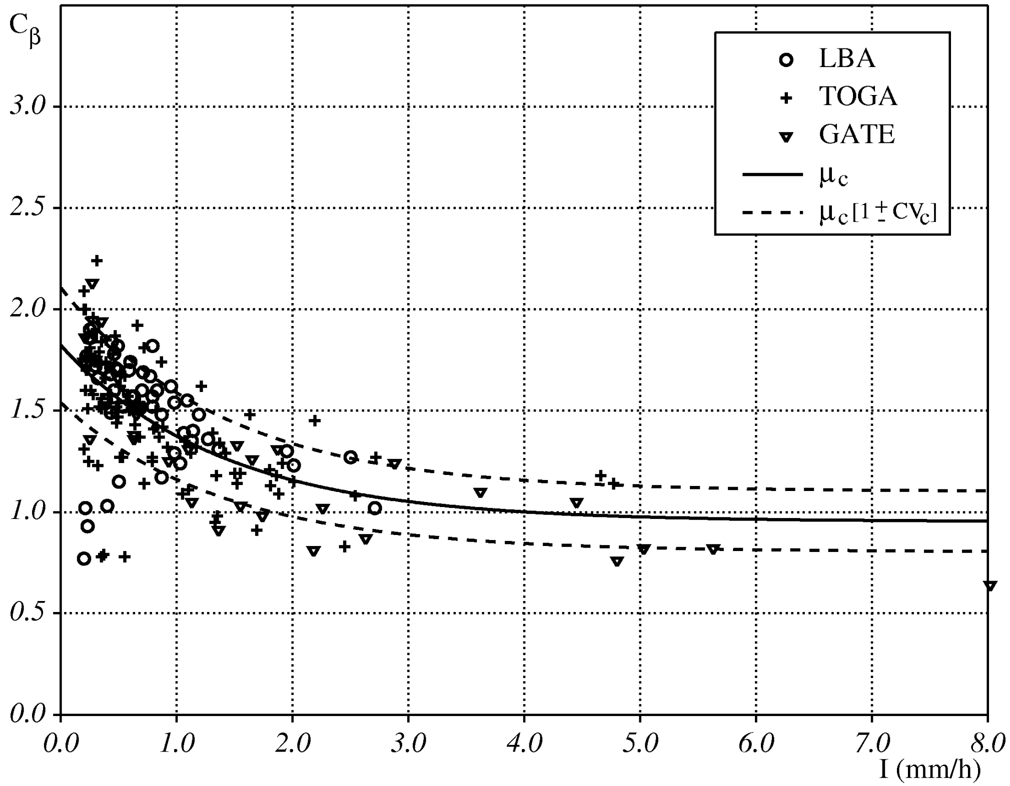
$$\zeta(q) = 3q + (c/\ln 2)\left[q(1-\beta) - (1-\beta^q)\right]$$

Estimates of c and β model parameters

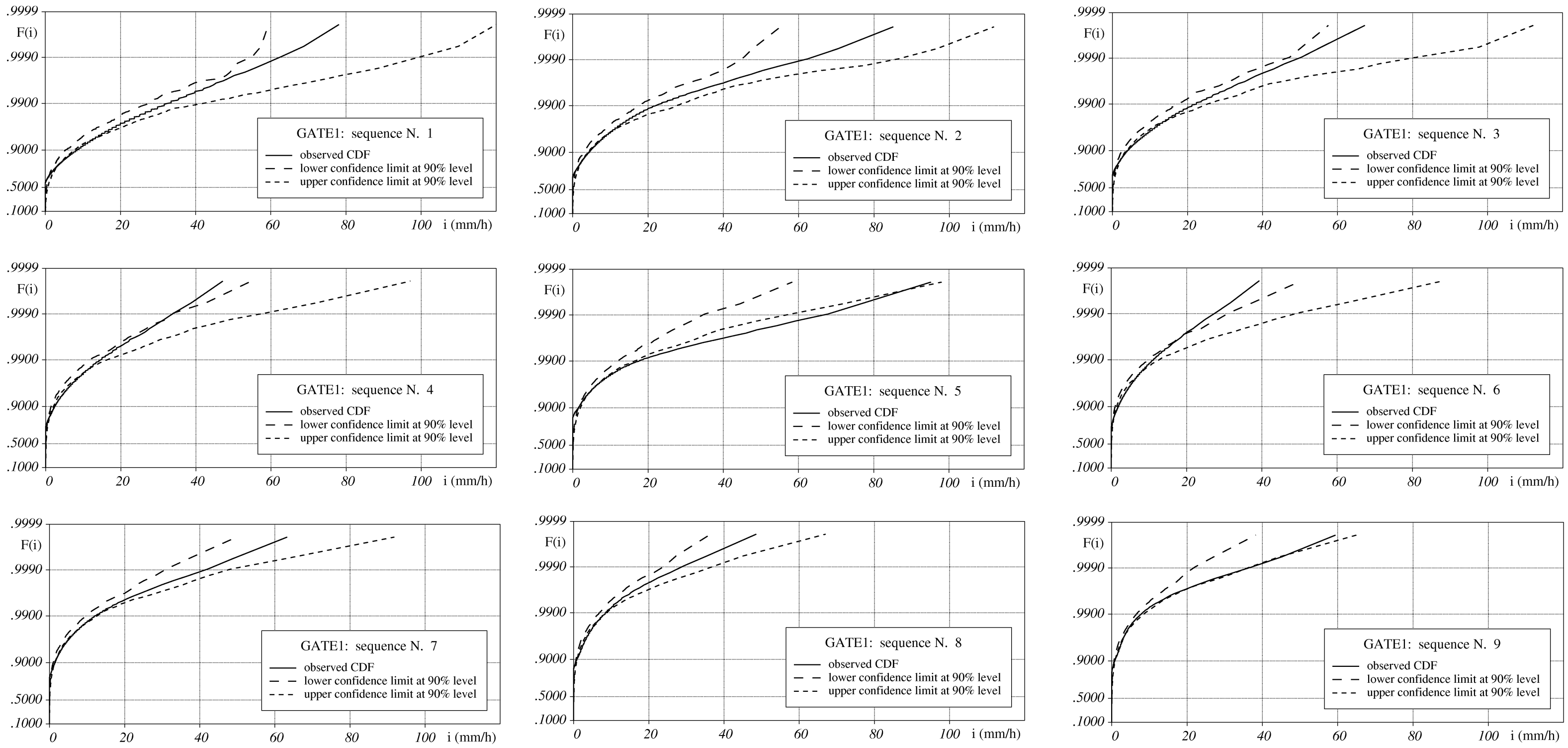
Estimates of β parameters are fairly constant around the mean value $\beta = 1/e$

Estimates of c parameters seem to be related to **large scale rain intensity I**:

$$c = c_0 \exp(-\gamma I) + c_\infty$$



GATE: CDF of small scale rain rate *i*

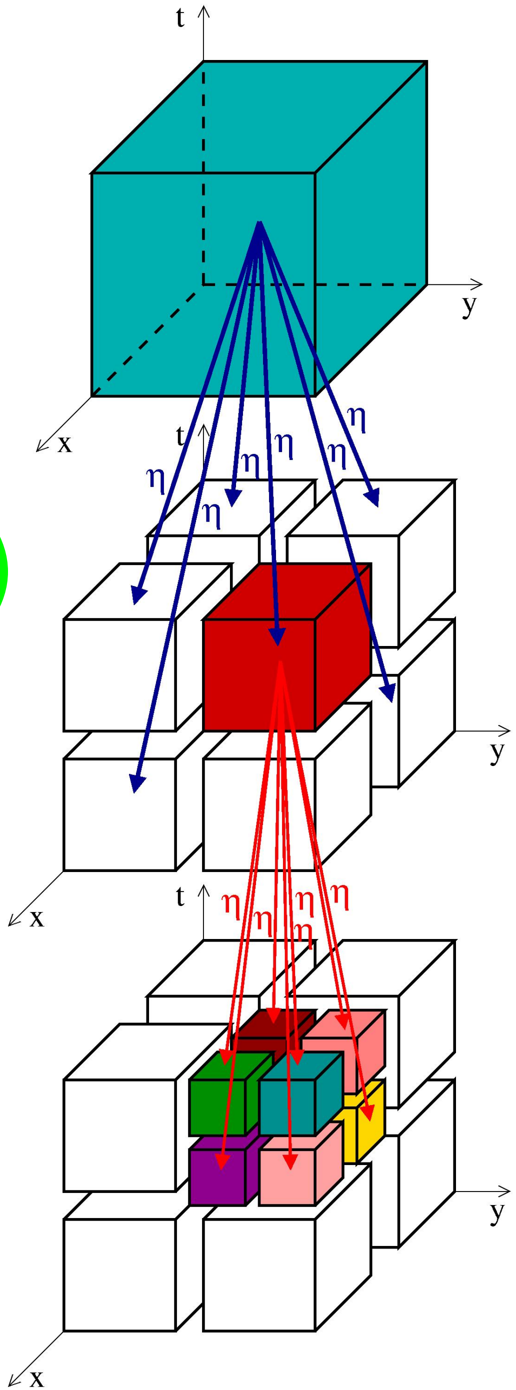


Cumulative Distribution Functions of small scale rainfall intensity (resolution 4 km and 15 minutes) are plotted with solid lines. Dashed lines represent the 90% confidence range

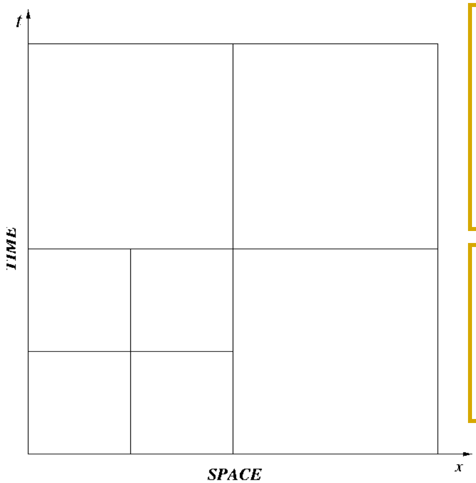
Two questions in space-time rainfall downscaling problems

- *Is there a relationship between space and time scales where we can observe the same statistical properties?*
  Self-similarity (i.e. scale isotropy)
  or
  Self-affinity (i.e. scale anisotropy)

- *Is the probability distribution of rain rate the same in each point/grid-cell (x,y)?*
  Space homogeneity
  or
  heterogeneity (e.g., due to orography)



Self-similarity vs Self-affinity
(scale isotropy) (scale anisotropy)



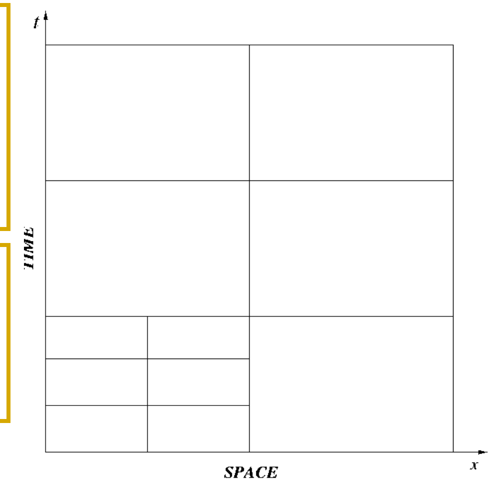
G.S.I. – Generalized Scale Invariance
(Lovejoy & Schertzer, 1985)

Scale changing operator T_b
(scaling anisotropy exponent H)

$$\begin{cases} \lambda \longrightarrow \lambda/b \\ \tau \longrightarrow \tau/b^{(1-H)} \end{cases}$$

Dynamic Scaling
(Kardar & al, 86; Czirok & al, 93; Venugopal et al, 99)

$$\tau = \text{const} \cdot \lambda^{Z}$$

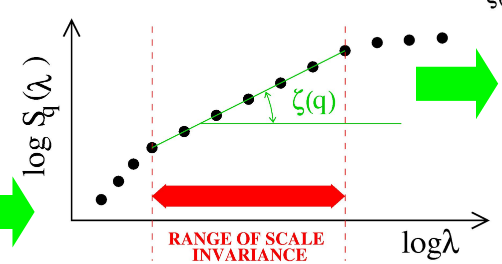


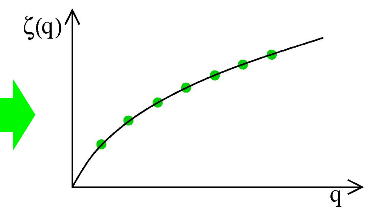
A reconjunction: → **Z = 1-H**

$H = 0$	(scaling anisotropy exponent)	$H \neq 0$
$Z = 1$	(dinamic scaling exponent)	$Z \neq 1$
$b_x = b_t$	(branching number)	$b_t = b_x^{(1-H)}$
$U = \lambda / \tau = \text{const}$	(ratio between space and time scales)	$U = U(\lambda) = U_L(\lambda/L)^H$

$$\mu_{i,j,k}(\lambda) = \int_{x_i}^{x_i+\lambda} dx \int_{y_j}^{y_j+\lambda} dy \int_{t_k}^{t_k+\lambda/U} dt \; i(x,y,t)$$

$$S_q(\lambda) = \left\langle \mu_{i,j,k}(\lambda)^q \right\rangle \approx \lambda^{\zeta(q)}$$





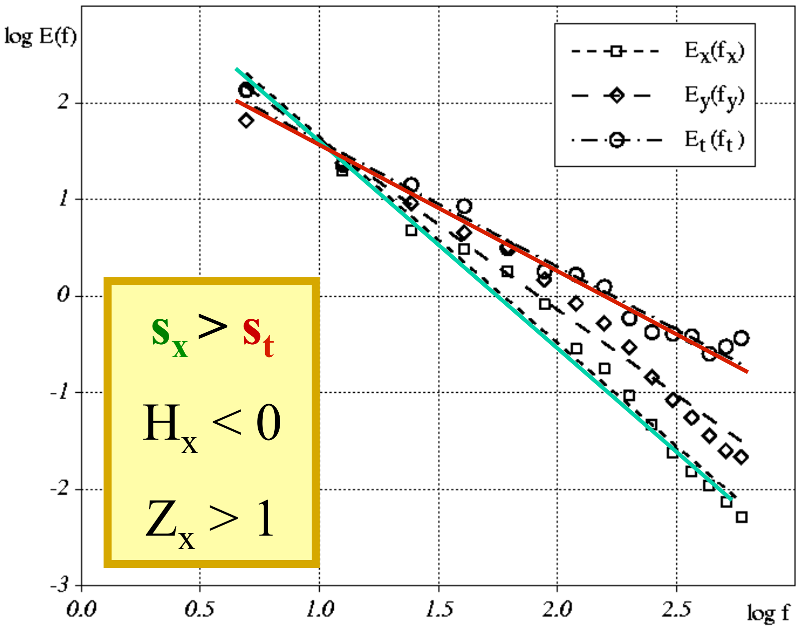
Self-similarity vs Self-affinity
(scale isotropy) (scale anisotropy)

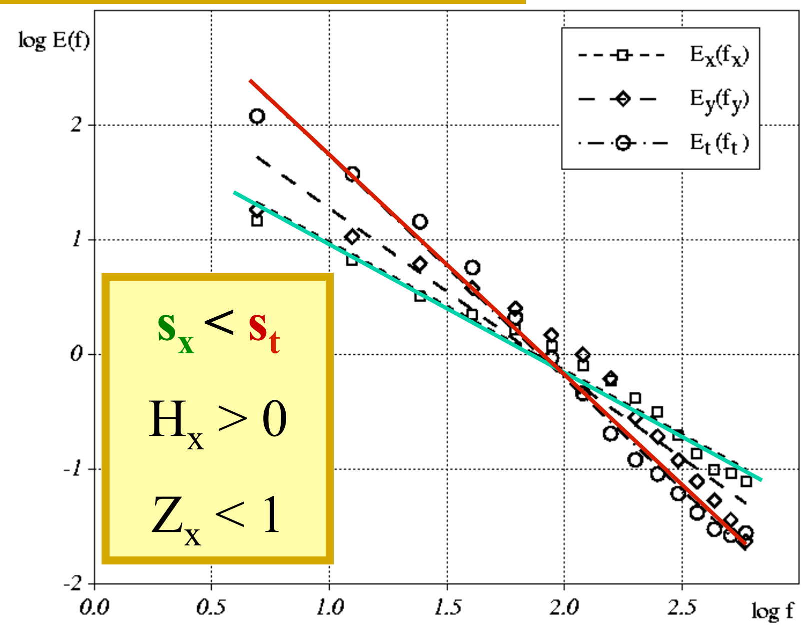
Power spectra of MF are power laws of frequency f_t and wave-numbers f_x, f_y :

$$E(f_t) \approx f_t^{-s_t} \qquad E(f_x) \approx f_x^{-s_x} \qquad E(f_y) \approx f_y^{-s_y}$$

Estimates of **H**: $H_x = 1 - s_x/s_t$ or $H_y = 1 - s_y/s_t$

Estimates of **Z**: $Z_x = s_x/s_t$ or $Z_y = s_y/s_t$



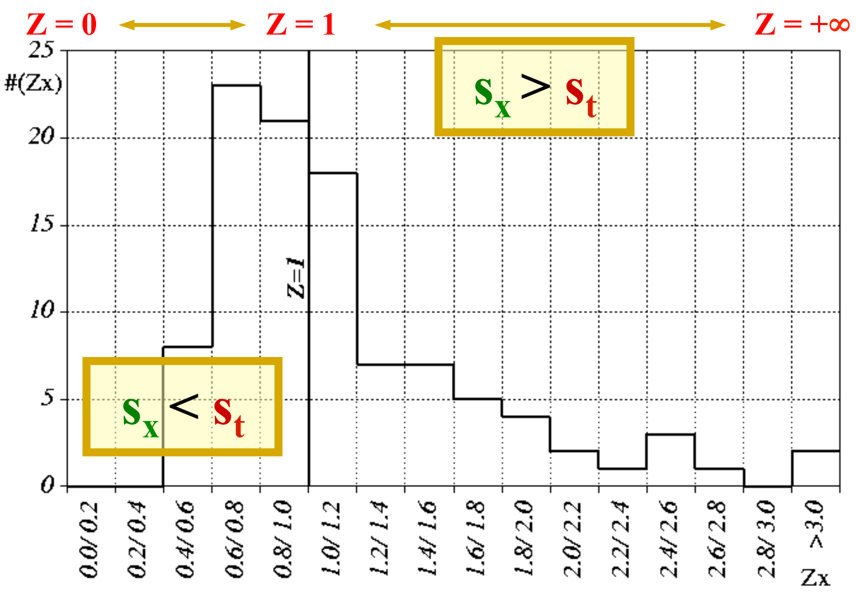


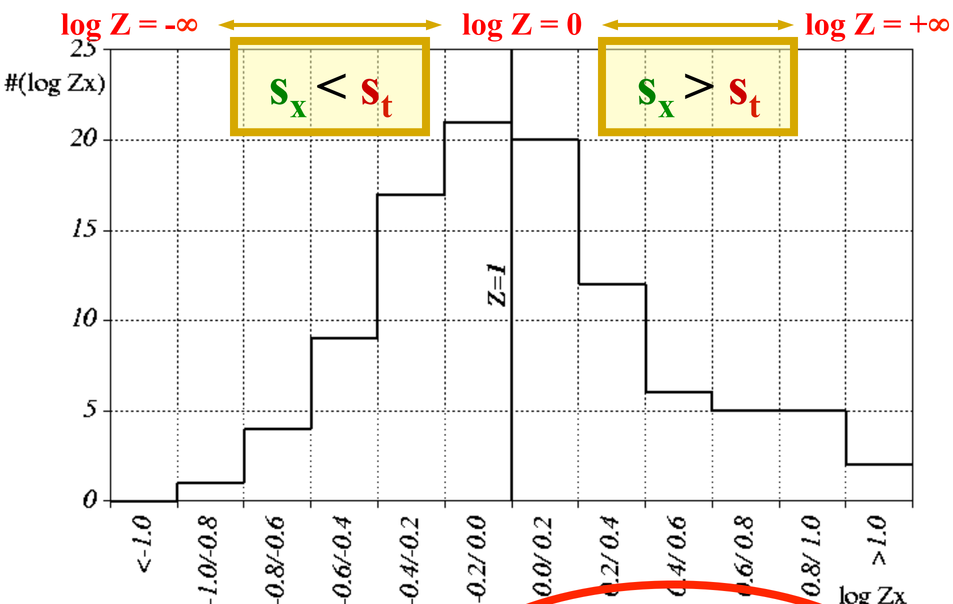
For self-similar measures we expect $H = 0$ or $Z = 1$

Self-similarity vs Self-affinity

Dynamic scaling exponents Z estimated on 102 TOGA-COARE events

$Z_x = s_x/s_t$ or $Z_y = s_y/s_t$





s_x	s_y	s_t	H_x	H_y	Z_x	Z_y	**$LogZ_x$**	**$LogZ_y$**
1.38	1.31	1.37	-0.21	-0.20	1.21	1.20	**0.05**	**0.00**

For self-similar measures we expect H = 0 or Z = 1 (i.e. log Z = 0)

More details in: Deidda, Badas, Piga (2004). Space-time scaling in high intensity TOGA-COARE storms, *Water Resources Research*, **40**

Two questions in space-time rainfall downscaling problems

- *Is there a relationship between space and time scales where we can observe the same statistical properties?*

  Self-similarity (i.e. scale isotropy)

  or

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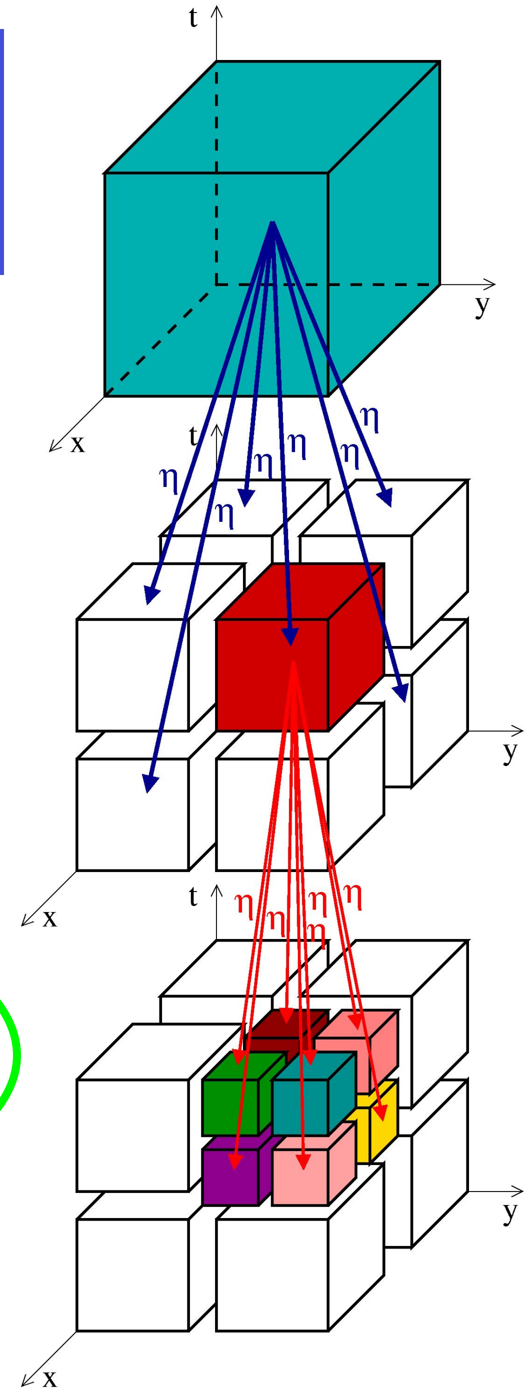
- *Is the probability distribution of rain rate the same in each point/grid-cell (x,y)?*

  Space homogeneity

  or

  heterogeneity (e.g., due to orography)



Second question:
Is the probability distribution of rain rate the same in each point/grid-cell (x,y)?

- ***Yes: spatial homogeneity***
  (examples: oceanic rainfall, such as GATE, TOGA-COARE)
  Multifractal models based on cascades with i.i.d. random generators (like the STRAIN model) can be applied.

- ***No, weak spatial heterogeneity*** that is **only** due to a different average of rainfall intensity from point to point:
  We can multiply a random cascade by a modulating function ξ(x,y)

$$\xi(x,y) = \overline{i(x,y,t)}$$

- ***No, strong spatial heterogeneity:*** the multifractal behaviour changes locally.
  The i.i.d. hypothesis for the random generator η cannot be assumed.

Rain rate modulating function $\xi(x,y)$

$$\xi(x,y) = \frac{\frac{1}{T}\int_0^T i(x,y,t)dt}{I}$$

I = large scale mean rain rate
(average on a time period T = 6 hours, and on a regional domain in space)

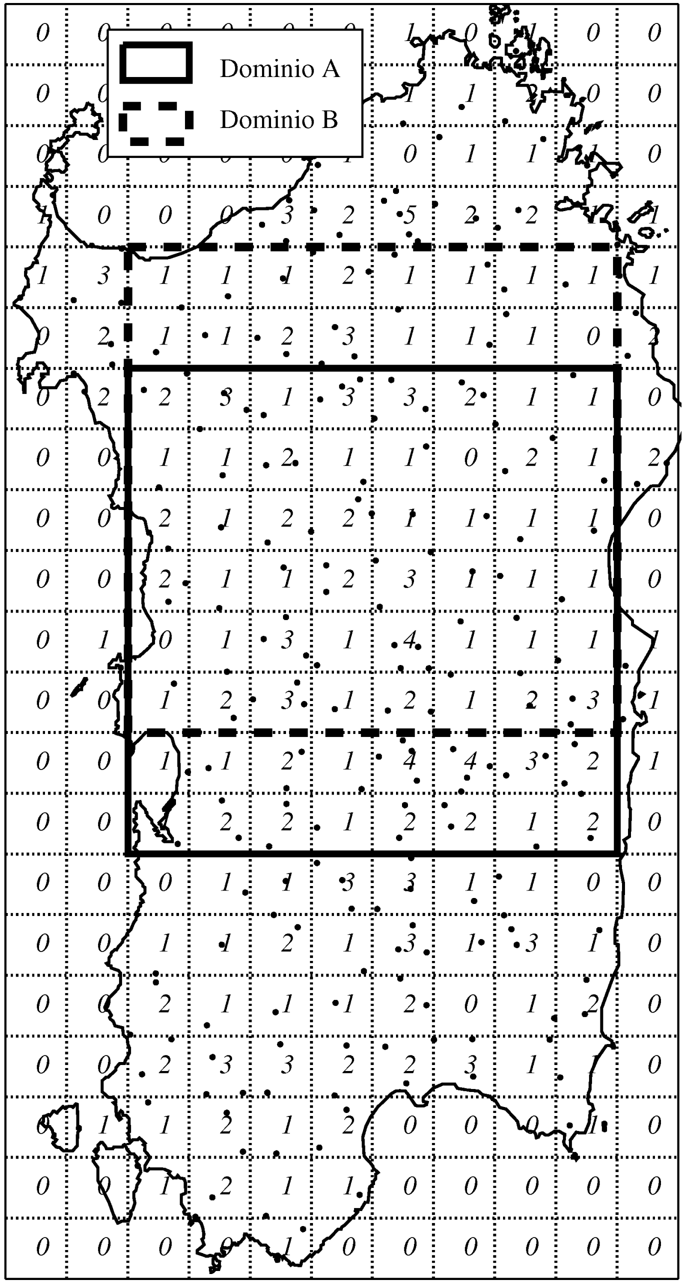
To filter out single event variability, we assume as modulating function the average <ξ> on a great number of events.

A **locally conditioned field** is thus:

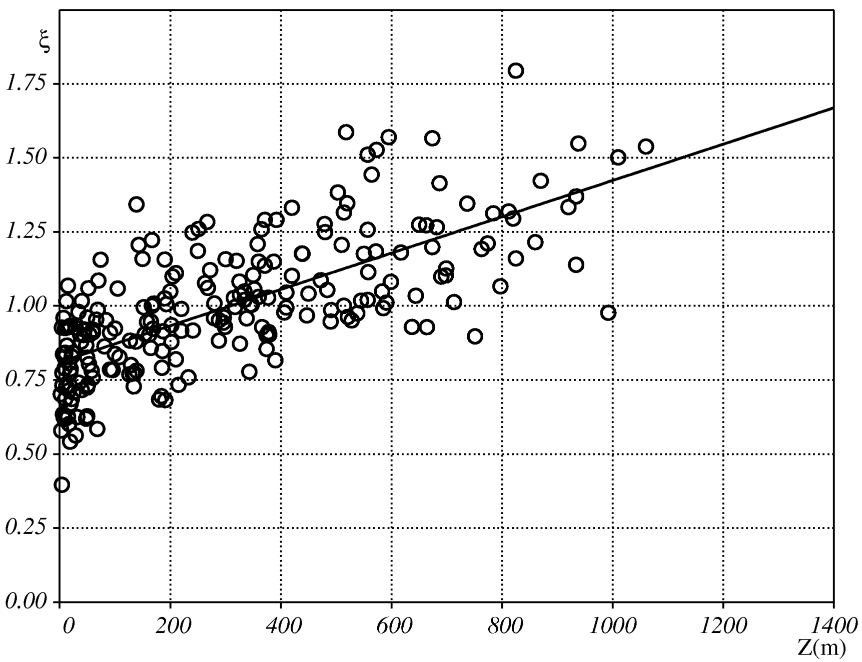
$$i(x,y,t) = \langle\xi(x,y)\rangle\, i_0(x,y,t)$$

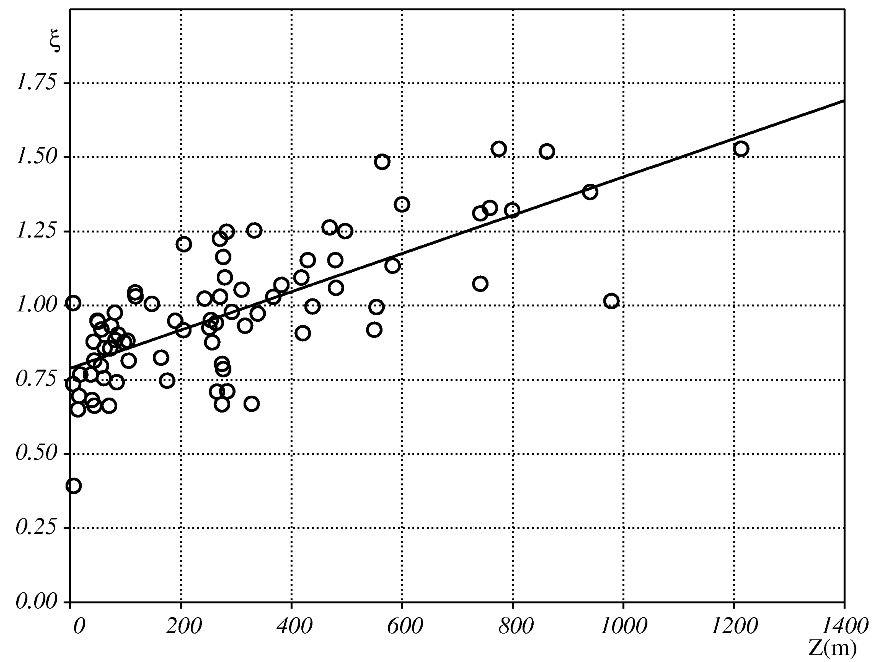
where i_0 is homogeneous in space.

- Analysis on single rain-gauge signals
- Analysis on gridded rain-gauges (L=103km)



Estimates of modulating function ξ(x,y)



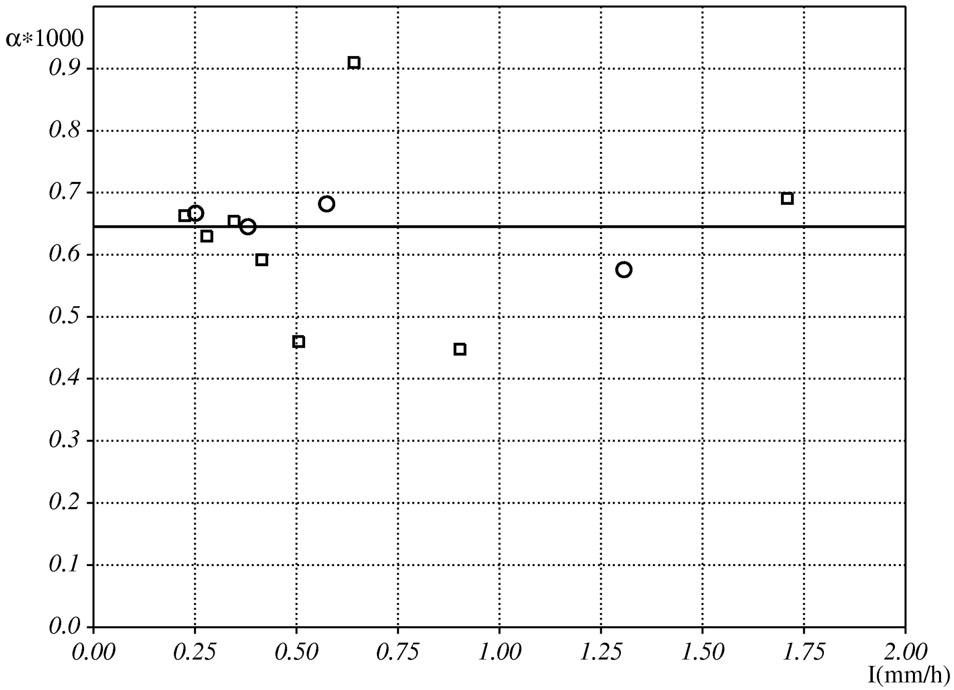


Rain-gauges: $\alpha = 0.61/1000$
794 events with duration T=6 hours

Gridded rain-gauges: $\alpha = 0.65/1000$
806 events with duration T=6 hours

$$\overline{\xi}(x,y) = \alpha \; z(x,y) + b \qquad \text{where } b = 1 - \alpha < z >_{x,y}$$

A sensitivity analysis on the slopes α

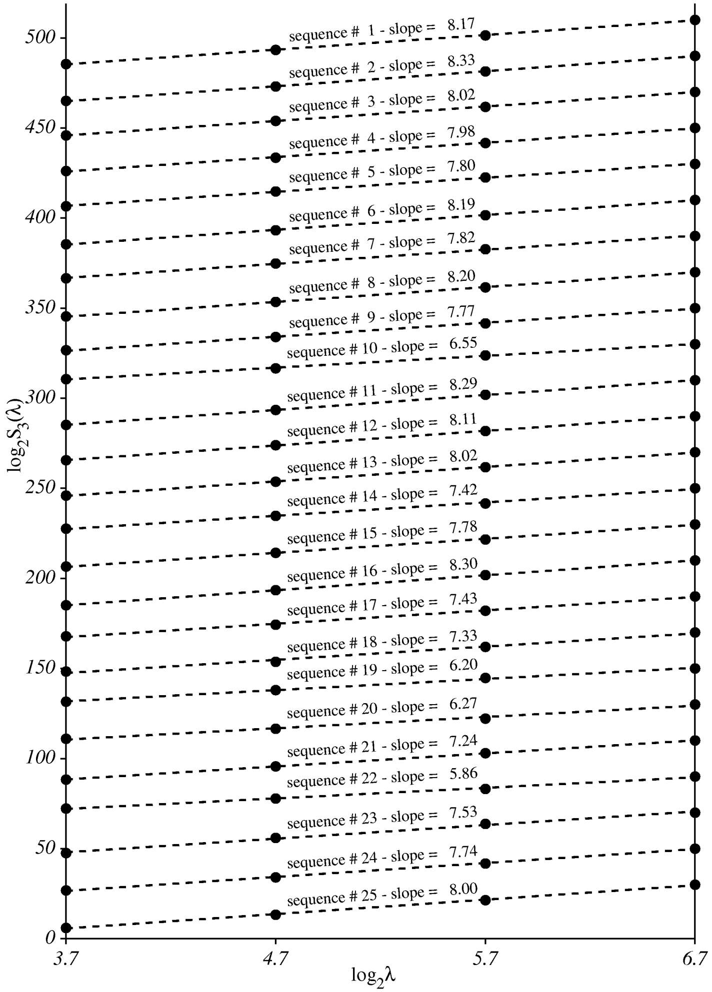


Estimates of α slopes versus mean large-scale rain rate I in classes of events

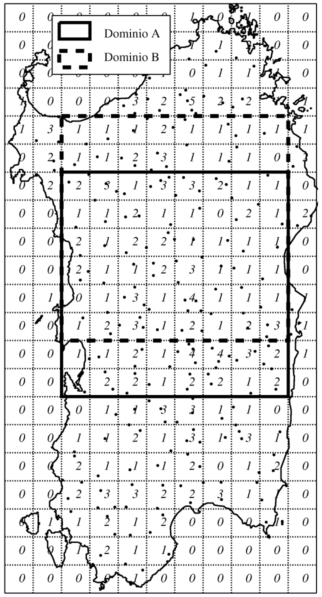
- *Continuous line*: all the 806 events
- *Circles:* 4 classes of about 200 events
- *Squares:* 8 classes of about 100 events

α slopes are independent on the large-scale rain rate I
Large variability in the estimates of α in small classes (squares)

MF analysis on gridded rain-gauges



$$S_q(\lambda) = \left\langle \mu_{i,j,k}(\lambda)^q \right\rangle \approx \lambda^{\zeta(q)}$$

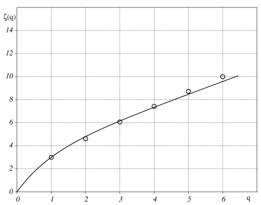


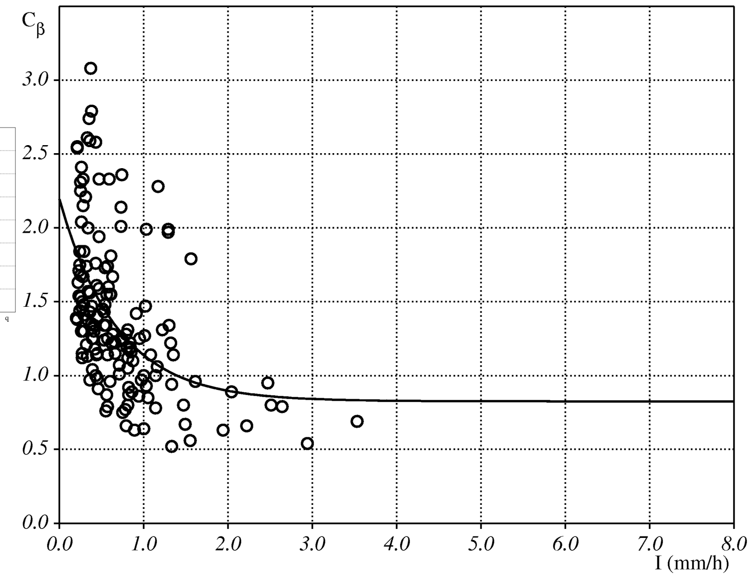
Scale invariance analysis:

λ = 13 km ÷104 km

τ = 45 min ÷ 6 hours

- 70 events on A domain
- 68 events on B domain





More details in: Badas, Deidda, Piga (2006). Modulation of homogeneous space-time rainfall cascades to account for orographic influences, *Natural Hazards and Earth System Sciences*, **6**.

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International Winter School on Hydrology - 2019 Edition

Doctoral Winter School DATA RICH HYDROLOGY

Modelling **scaling** properties of precipitation fields

Thanks for your attention

Roberto Deidda

Dipartimento di Ingegneria Civile, Ambientale ed Architettura

Università di Cagliari

Perugia (Italy), January 28 – February 1, 2019 – Villa Colombella